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RIEMANN PROBLEM FOR THE ONE-VELOCITY MODEL OF A HETEROGENEOUS MEDIUM

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The internal interfractional-interaction forces have been allowed for in the one-velocity model of a heterogeneous medium, and a complete solution of the Riemann problem has been obtained. The shock adiabat of the mixture, consistent with the model's equations, has been used in constructing the solution with shock waves.

Introduction. The one-velocity model of a heterogeneous medium finds extensive use in investigating wave phenomena in gas-liquid media of a foam structure, bubble liquids, porous materials (of the foam-polyurethane type), and in other heterogeneous systems where the one-velocity approximation is acceptable. We know of two modifications of the one-velocity model: in the first modification, the action of internal interfractional-interaction forces has been disregarded (see [1]), whereas in the second, conversely, these forces have been allowed for (see [2]). The solution of the Riemann problem, or the problem on disintegration of an arbitrary discontinuity in a medium representing a mixture of different generally compressible components, has been given in [3] for the modification from [1]. This solution was obtained owing to the presence of the analytical expression for the mixture's isentrope used in calculations of Riemann invariants, when rarefaction waves are computed. No analytical expression for the isentrope of the mixture exists for the variant of the medium's model from [2]; therefore, we are unable to use the theory of Riemann invariants here. Nonetheless, the Riemann problem can be solved. In the present work, we give the solution of the problem on disintegration of an arbitrary discontinuity in a multicomponent mixture, in which internal interfractional-interaction forces are allowed for. In calculations of flows in rarefaction waves, instead of solving nonlinear algebraic equations (to which the problem in [3] was reduced), we have to integrate systems of ordinary differential equations. In constructing solutions with shock waves, we use, unlike [3], the shock adiabat of the mixture from [4], which is consistent with the equations of the one-velocity model. The solution obtained in the work is of both theoretical and practical importance, since the algorithm of the Riemann problem appears as the basic element of the code in computer programs in which numerical schemes of the S. K. Godunov type are used (see [5]).

When the behavior of the mixture's components is described, for the sake of definiteness we use the equation of state

$$\varepsilon_i = \frac{p - c_{*i}^2 (\rho_i^0 - \rho_{*i})}{\rho_i^0 (\gamma_i - 1)} = \frac{b_i + pB_i}{\rho_i^0} - d_i \,, \tag{1}$$

where $B_i = 1/(\gamma_i - 1)$, $d_i = c_{*i}^2 B_i$, and $b_i = d_i \rho_{*i}$. When (1) is used, the equation of state of the n-component mixture with the first m compressible fractions takes the form (see [2])

$$\varepsilon = \frac{1}{\rho}\left[b_m + pB_m + \sum_{i=1}^{m-1} \alpha_i \left(b_{im} - d_{im}\rho_i^0 + pB_{im} \right) + \sum_{j=m+1}^{n} \alpha_j \rho_j^0 \varepsilon_j \right] - d_m \,. \tag{2}$$

Here $b_{im} = b_i - b_m$, $B_{im} = B_i - B_m$, and $d_{im} = d_i - d_m$.

Disintegration of an Arbitrary Discontinuity in a Multicomponent Mixture. Let two infinite masses of dispersive media, consisting of n_1 and n_2 components each, be located to the "left" of the plane $x = 0$ and to the "right"

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of it respectively at the initial instant of time ($t = 0$). The pressures, velocities, densities, and volume fractions of the mixture's components in the above media are constant and equal to $p_{0(1)}$, $u_{0(1)}$, $\rho_{0(1)}$, $\rho^0_{i0(1)}$, and $\alpha_{i0(1)}$ ($i = 1, ..., n_1$) and to $p_{0(2)}$, $u_{0(2)}$, $\rho_{0(2)}$, $\rho^0_{i0(2)}$, and $\alpha_{i0(2)}$ ($i = 1, ..., n_2$) respectively. Without loss of generality, we will assume that the pressure to the "left" of the contact discontinuity is no lower than that to its "right," i.e., $p_{0(1)} \geq p_{0(2)}$. If, conversely, we have $p_{0(1)} < p_{0(2)}$, then, reversing the direction of the $0x$ axis and redenoting the subscripts of the variables, we arrive at the previous case. It is necessary to calculate flow occurring for $t > 0$. We will seek the self-similar solution of the problem, i.e., such a solution where all dependent variables are functions of just one independent variable $\xi = x/t$. In particular, it follows that the discontinuity lines are straight lines in the (x, t) plane, i.e., the velocities of the shock waves and the contact discontinuity are constant. If the arbitrary discontinuity is neither contact-type nor a shock wave, it disintegrates, forming either two shock waves, or a rarefaction wave and a shock wave, or two rarefaction waves. We consider the above cases in detail.

Two Shock Waves. A system of equations for the "right-hand" shock wave, which expresses the laws of conservation of mass and momentum in passage through the shock-wave front, together with the equation of shock adiabat of the mixture (see [4]) has the form

$$D_2 (\rho_{+(2)} - \rho_{0(2)}) = \rho_{+(2)} u_{+(2)} - \rho_{0(2)} u_{0(2)} , \tag{3}$$

$$D_2 (\rho_{+(2)} u_{+(2)} - \rho_{0(2)} u_{0(2)}) = p_{+(2)} + \rho_{+(2)} u^2_{+(2)} - p_{0(2)} - \rho_{0(2)} u^2_{0(2)} . \tag{4}$$

We use the relation

$$\varepsilon_{+(2)} - \varepsilon_{0(2)} = \frac{p_{+(2)} + p_{0(2)}}{2} \left(\frac{1}{\rho_{0(2)}} - \frac{1}{\rho_{+(2)}} \right),$$

which, with account for (2), will be represented as

$$\frac{1}{\rho_{+(2)}} \left\{ \sum_{i=1}^{m_2-1} \alpha_{i+(2)} \left(b_{im_2} - d_{im_2} \rho^0_{i+(2)} + p_{+(2)} B_{im_2} \right) + b_{m_2(2)} + p_{+(2)} B_{m_2(2)} + \sum_{j=m_2+1}^{n_2} \alpha_{j+(2)} \rho^0_{j+(2)} \varepsilon_{j+(2)} \right\}$$

$$- \frac{1}{\rho_{0(2)}} \left\{ \sum_{i=1}^{m_2-1} \alpha_{i0(2)} \left(b_{im_2} - d_{im_2} \rho^0_{i0(2)} + p_{0(2)} B_{im_2} \right) + b_{m_2} + p_{0(2)} B_{m_2} + \sum_{j=m_2+1}^{n_2} \alpha_{j0(2)} \rho^0_{j0(2)} \varepsilon_{j0(2)} \right\}$$

$$= \frac{p_{+(2)} + p_{0(2)}}{2} \left(\frac{1}{\rho_{0(2)}} - \frac{1}{\rho_{+(2)}} \right). \tag{5}$$

If we substitute into relation (5) the expressions

$$\rho^0_{i+(2)} = \frac{b_{i(2)} + p_{+(2)} B_{i(2)} - \dfrac{u_{+(2)} p_{+(2)} (\rho_{+(2)} - \rho_{0(2)})}{\rho_{0(2)} (u_{+(2)} - u_{0(2)})}}{\dfrac{u^2_{0(2)}}{2} - \dfrac{u^2_{+(2)}}{2} + \dfrac{1}{\rho^0_{i0(2)}} \left[b_{i(2)} + p_{0(2)} B_{i(2)} - \dfrac{u_{0(2)} p_{0(2)} (\rho_{+(2)} - \rho_{0(2)})}{\rho_{+(2)} (u_{+(2)} - u_{0(2)})} \right]} ,$$

$$\alpha_{i+(2)} = \frac{\alpha_{i0(2)} \rho_{+(2)} \rho^0_{i0(2)} \left\{ \dfrac{u^2_{0(2)}}{2} - \dfrac{u^2_{+(2)}}{2} + \dfrac{1}{\rho^0_{i0(2)}} \left[b_{i(2)} + p_{0(2)} B_{i(2)} - \dfrac{u_{0(2)} p_{0(2)} (\rho_{+(2)} - \rho_{0(2)})}{\rho_{+(2)} (u_{+(2)} - u_{0(2)})} \right] \right\}}{\rho_{0(2)} \left[b_{i(2)} + p_{+(2)} B_{i(2)} - \dfrac{u_{+(2)} p_{+(2)} (\rho_{+(2)} - \rho_{0(2)})}{\rho_{0(2)} (u_{+(2)} - u_{0(2)})} \right]} , \tag{6}$$

$$u_{+(2)} = u_{0(2)} + \sqrt{(p_{+(2)} - p_{0(2)})\left(\frac{1}{\rho_{0(2)}} - \frac{1}{\rho_{+(2)}}\right)}\ , \quad i = 1, ..., m_2 - 1\ ;$$

$$\alpha_{j+(2)} = \alpha_{j0(2)} \frac{\rho_{+(2)}}{\rho_{0(2)}}\ , \quad j = m_2 + 1, ..., n_2\ ,$$

which are a consequence of the laws of conservation of the mass, momentum, and energy fluxes for the constituent fractions of the mixture, we obtain the shock adiabat of the mixture $H_{(2)}(p_{+(2)}, \rho_{+(2)}) = 0$, whence we have $\rho_{+(2)} = h_{(2)}(p_{+(2)})$. The relation for the velocity $u_{+(2)}$, which is expressed by the remaining parameters of flow, has the form

$$u_{+(2)} = u_{0(2)} + f_{(2)}\,(p_{+(2)})\ , \tag{7}$$

where

$$f_{(2)}\,(p_{+(2)}) = \sqrt{(p_{+(2)} - p_{0(2)})\left(\frac{1}{\rho_{0(2)}} - \frac{1}{h_{(2)}p_{+(2)}}\right)}\ . \tag{8}$$

For the "left-hand" shock wave, we have an analogous system of equations:

$$D_1\left(\rho_{-(1)} - \rho_{0(1)}\right) = \rho_{-(1)}u_{-(1)} - \rho_{0(1)}u_{0(1)}\ , \quad D_1\left(\rho_{-(1)}u_{-(1)} - \rho_{(1)}u_{0(1)}\right) = p_{-(1)} + \rho_{-(1)}u_{-(1)}^2 - p_{0(1)} - \rho_{0(1)}u_{0(1)}^2\ ; \tag{9}$$

$$\frac{1}{\rho_{-(1)}}\left\{\sum_{i=1}^{m_1-1}\alpha_{i-(1)}\left(b_{im_1} - d_{im_1}\overset{0}{\rho}_{i-(1)} + p_{-(1)}B_{im_1}\right) + b_{m_1} + p_{-(1)}B_{m_1} + \sum_{j=m_1+1}^{n_1}\alpha_{j-(1)}\overset{0}{\rho}_{j-(1)}\varepsilon_{j-(1)}\right\}$$

$$-\frac{1}{\rho_{0(1)}}\left\{\sum_{i=1}^{m_1-1}\alpha_{i0(1)}\left(b_{im_1} - d_{im_1}\overset{0}{\rho}_{i0(1)} + p_{0(1)}B_{im_1}\right) + b_{m_1} + p_{0(1)}B_{m_1} + \sum_{j=m_1+1}^{n_1}\alpha_{j0(1)}\overset{0}{\rho}_{j0(1)}\varepsilon_{j0(1)}\right\}$$

$$= \frac{p_{-(1)} + p_{0(1)}}{2}\left(\frac{1}{\rho_{0(1)}} - \frac{1}{\rho_{-(1)}}\right). \tag{10}$$

To obtain the shock adiabat $H_{(1)}(p_{-(1)}, \rho_{-(1)}) = 0$ we must substitute into relation (10) the expressions

$$\overset{0}{\rho}_{i-(1)} = \frac{b_{i(1)} + p_{-(1)}B_{i(1)} - \dfrac{u_{-(1)}p_{-(1)}\,(\rho_{-(1)} - \rho_{0(1)})}{\rho_{0(1)}\,(u_{-(1)} - u_{0(1)})}}{\dfrac{u_{0(1)}^2}{2} - \dfrac{u_{-(1)}^2}{2} + \dfrac{1}{\overset{0}{\rho}_{i0(1)}}\left[b_{i(1)} + p_{0(1)}B_{i(1)} - \dfrac{u_{0(1)}p_{0(1)}\,(\rho_{-(1)} - \rho_{0(1)})}{\rho_{-(1)}\,(u_{-(1)} - u_{0(1)})}\right]}\ ,$$

$$\alpha_{i-(1)} = \frac{\alpha_{i0(1)}\rho_{-(1)}\overset{0}{\rho}_{i0(1)}\left\{\dfrac{u_{0(1)}^2}{2} - \dfrac{u_{-(1)}^2}{2} + \dfrac{1}{\overset{0}{\rho}_{i0(1)}}\left[b_{i(1)} + p_{0(1)}B_{i(1)} - \dfrac{u_{0(1)}p_{0(1)}\,(\rho_{-(1)} - \rho_{0(1)})}{\rho_{-(1)}\,(u_{-(1)} - u_{0(1)})}\right]\right\}}{\rho_{0(1)}\left[b_{i(1)} + p_{-(1)}B_{i(1)} - \dfrac{u_{-(1)}p_{-(1)}\,(\rho_{-(1)} - \rho_{0(1)})}{\rho_{0(1)}\,(u_{-(1)} - u_{0(1)})}\right]}\ , \tag{11}$$

$$u_{-(1)} = u_{0(1)} - \sqrt{\left(p_{-(1)} - p_{0(1)}\right)\left(\frac{1}{\rho_{0(1)}} - \frac{1}{\rho_{-(1)}}\right)}\ , \quad i = 1, ..., m_1 - 1\ ;$$

$$\alpha_{j-(1)} = \alpha_{j0(1)} \frac{\rho_{-(1)}}{\rho_{0(1)}}\ , \quad j = m_1 + 1, ..., n_1\ ,$$

whence we have $\rho_{-(1)} = h_{(1)}(p_{-(1)})$. Consequently, we obtain

$$u_{-(1)} = u_{0(1)} - f_{(1)}\,(p_{-(1)})\ , \tag{12}$$

where

$$f_{(1)}\,(p_{-(1)}) = \sqrt{\left(p_{-(1)} - p_{0(1)}\right)\left(\frac{1}{\rho_{0(1)}} - \frac{1}{h_{(1)}\rho_{-(1)}}\right)}\ . \tag{13}$$

Pressure and velocity at the contact boundary undergo no changes; therefore, the conjugation conditions

$$p_{+(2)} = p_{-(1)} = P\ , \quad u_{+(2)} = u_{-(1)} = U \tag{14}$$

are observed.
Relations (7) and (12), with account for (14), yield an equation for the pressure on the contact discontinuity P:

$$u_{0(1)} - f_{(1)}\,(P) = u_{0(2)} + f_{(2)}\,(P)\ , \tag{15}$$

where $f_{(1)}(P)$ and $f_{(2)}(P)$ are determined from expressions (8) and (13). We rewrite (15) in the form

$$u_{0(1)} - u_{0(2)} = f_{(1)}\,(P) + f_{(2)}\,(P)\ . \tag{16}$$

Since the right-hand side of relation (16) is an increasing function of its argument P, its minimum value is attained for $P = p_{0(1)}$. We note that the radicand of the function $f_{(1)}(P)$ becomes negative when $P < p_{0(1)}$ and Eq. (15) loses its meaning. Let us denote this minimum value taken by the right-hand side of (16) by U_*. Allowing for expression (13), we represent U_* in the form

$$U_* = f_{(2)}\,(p_{0(1)})\ . \tag{17}$$

Thus, the configuration with two shock waves occurs where the inequality

$$u_{0(1)} - u_{0(2)} \geq U_*\ . \tag{18}$$

holds. Otherwise, this regime of flow is impossible. The sought root of Eq. (15) is calculated numerically with the standard procedure of solution of nonlinear equations; its value is always higher than the initial pressure $p_{0(1)}$.

As an example of using the algorithm of calculation of the disintegration of an arbitrary discontinuity, we consider the problem on collision ($u_{0(2)} = -u_{0(1)}$) of two identical flows of gas-liquid mixtures ($\gamma_{1(1)} = \gamma_{1(2)} = 1.4$, $c_{*1(1)} = c_{*1(2)} = 0$, $\rho_{*1(2)} = \rho_{*1(1)} = 1.19$ kg/m^3, $\gamma_{2(1)} = \gamma_{2(2)} = 5.59$, $c_{*2(1)} = c_{*2(2)} = 1.500$ m/sec, and $\rho_{*2(1)} = \rho_{*2(2)} = 1.000$ kg/m^3). The interaction of these flows produces symmetric flow with two shock waves. The collision velocities are varied from 3 to 200 m/sec. Figure 1 gives the dependences of the distributions of the relative pressure in the shock wave $P/p_{0(1)}(u_{0(1)})$ and the velocity of movement of the shock-wave front $D(u_{0(1)})$ in the gas-liquid mixture for $p_{0(1)} = p_{0(2)} = 10^5$ Pa and $\alpha_{10(1)} = \alpha_{10(2)} = 0.8$. For the sake of comparison this figure gives analogous dependences $P/p_{0(1)}(u_{0(1)})$ and $D(u_{0(1)})$ (dashed curves) obtained by solution of the problem on disintegration of an

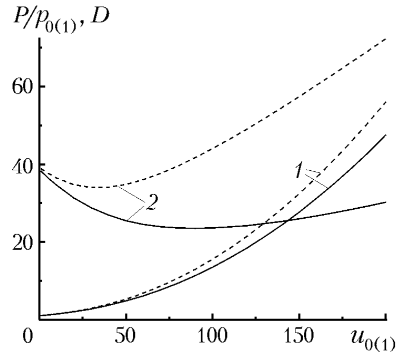

Fig. 1. Dependences of the parameters of flow with two shock waves: 1) $P/p_{0(1)}(u_{0(1)})$ and 2) $D(u_{0(1)})$. The dashed curves are calculated from the algorithm from [3]).

arbitrary discontinuity according to the algorithm from [3]; in the problem, the assumption of the compression of the mixture's components by individual shock adiabats was used.

Rarefaction Wave and Shock Wave. We consider another possible type of discontinuity disintegration that includes the centered rarefaction wave propagating to the left of the contact discontinuity apart from the shock wave.

Relations (3)–(6) are true for the shock wave. In calculating the rarefaction wave propagating in the n_1th-component mixture with m_1 compressible fractions, we must integrate the following system of partial equations (see [2]):

$$\frac{\partial \rho_{(1)}}{\partial t} + u_{(1)} \frac{\partial \rho_{(1)}}{\partial x} + \rho_{(1)} \frac{\partial u_{(1)}}{\partial x} = 0\,, \quad \frac{\partial u_{(1)}}{\partial t} + u_{(1)} \frac{\partial u_{(1)}}{\partial x} + \frac{1}{\rho_{(1)}} \frac{\partial p_{(1)}}{\partial x} = 0\,,$$

$$C_{1(1)} \left(\frac{\partial \rho_{(1)}}{\partial t} + u_{(1)} \frac{\partial \rho_{(1)}}{\partial x} \right) + C_{2(1)} \left(\frac{\partial p_{(1)}}{\partial t} + u_{(1)} \frac{\partial p_{(1)}}{\partial x} \right)$$

$$+ \sum_{i=1}^{m_1-1} \left[C_{1+2(1)} \left(\frac{\partial \overset{0}{\rho}_{i(1)}}{\partial t} + u_{(1)} \frac{\partial \overset{0}{\rho}_{i(1)}}{\partial x} \right) + C_{i+m_1+1(1)} \left(\frac{\partial \alpha_{i(1)}}{\partial t} + u_{(1)} \frac{\partial \alpha_{i(1)}}{\partial x} \right) \right]$$

$$+ \sum_{j=m_1+1}^{n_1} C_{j+m_1(1)} \left(\frac{\partial \alpha_{j(1)}}{\partial t} + u_{(1)} \frac{\partial \alpha_{j(1)}}{\partial x} \right) = 0\,, \tag{19}$$

$$-\frac{1}{\rho_{(1)}} \left(\frac{\partial \rho_{(1)}}{\partial t} + u_{(1)} \frac{\partial \rho_{(1)}}{\partial x} \right) + \frac{1}{\overset{0}{\rho}_{i(1)}} \left(\frac{\partial \overset{0}{\rho}_{i(1)}}{\partial t} + u_{(1)} \frac{\partial \overset{0}{\rho}_{i(1)}}{\partial x} \right) + \frac{1}{\alpha_{i(1)}} \left(\frac{\partial \alpha_{i(1)}}{\partial t} + u_{(1)} \frac{\partial \alpha_{i(1)}}{\partial x} \right) = 0\,,$$

$$\frac{B_{i(1)}}{p_{(1)}} \left(\frac{\partial p_{(1)}}{\partial t} + u_{(1)} \frac{\partial p_{(1)}}{\partial x} \right) - \frac{b_{i(1)} + p_{(1)} (1 + B_{i(1)})}{p_{(1)} \overset{0}{\rho}_{i(1)}} \left(\frac{\partial \overset{0}{\rho}_{i(1)}}{\partial t} + u_{(1)} \frac{\partial \overset{0}{\rho}_{i(1)}}{\partial x} \right)$$

$$-\frac{1}{\alpha_{i(1)}} \left(\frac{\partial \alpha_{i(1)}}{\partial t} + u_{(1)} \frac{\partial \alpha_{i(1)}}{\partial x} \right) = 0\,, \quad i = 1, ..., m_1 - 1\,;$$

$$\frac{\partial \alpha_{j(1)}}{\partial t} + u_{(1)} \frac{\partial \alpha_{j(1)}}{\partial x} + \alpha_{j(1)} \frac{\partial u_{(1)}}{\partial x} = 0\,, \quad j = m_1 + 1, ..., n_1\,.$$

Here we have used the notation

$$C_{1(1)} = -\frac{1}{\rho_{(1)}}\left[b_{m_1} + p_{(1)}B_{m_1} + \sum_{i=1}^{m_1-1}\alpha_{i(1)}\left(b_{im_1} - d_{im_1}\overset{0}{\rho}_{i(1)} + p_{(1)}B_{im_1}\right) + \sum_{j=m_1+1}^{n_1}\alpha_{j(1)}A_{j+m_1(1)}\right],$$

$$C_{2\,(1)} = B_{m_1} + \sum_{i=1}^{m_1-1}\alpha_{i(1)}B_{im_1},$$

$$C_{i+2(1)} = -\,d_{im_1}\alpha_{i(1)}, \quad C_{i+m_1+1(1)} = b_{im_1} - \overset{0}{\rho}_{i(1)}d_{im_1} + p_{(1)}B_{im_1}, \quad i = 1, ..., m_1 - 1;$$

$$C_{j+m_1(1)} = \overset{0}{\rho}_{j(1)}\varepsilon_{j(1)} = \text{const}, \quad j = m_1 + 1, ..., n_1.$$

On introduction of the self-similar variable $\xi = x/t$, the system of equations (19) is transformed to a system of ordinary differential equations (see [6]):

$$\frac{dp_{(1)}}{d\xi} = -\frac{2\rho_{(1)}\,(u_{(1)} - \xi)^3}{(u_{(1)} - \xi)^2\,(2 + \rho_{(1)}Q_{1(1)}) + \rho_{(1)}Q_{2(1)}}, \quad \frac{du_{(1)}}{d\xi} = \frac{2\,(u_{(1)} - \xi)^2}{(u_{(1)} - \xi)^2\,(2 + \rho_{(1)}Q_{1(1)}) + \rho_{(1)}Q_{2(1)}},$$

$$\frac{d\rho_{(1)}}{d\xi} = -\frac{2\rho_{(1)}\,(u_{(1)} - \xi)}{(u_{(1)} - \xi)^2\,(2 + \rho_{(1)}Q_{1(1)}) + \rho_{(1)}Q_{2(1)}},$$

$$\frac{d\overset{0}{\rho}_{i(1)}}{d\xi} = -\frac{2\overset{0}{\rho}_{i(1)}\,(u_{(1)} - \xi)\,[\rho_{(1)}B_{i(1)}\,(u_{(1)} - \xi)^2 - p_{(1)}]}{(b_{i(1)} + p_{(1)}B_{i(1)})\,[(u_{(1)} - \xi)^2\,(2 + \rho_{(1)}Q_{1(1)}) + \rho_{(1)}Q_{2(1)}]}, \quad i = 1, ..., m_1 - 1. \tag{20}$$

Here we have

$$Q_{1(1)} = \frac{\partial G_{(1)}}{\partial p_{(1)}} + \sum_{i=1}^{m_1-1}\frac{\overset{0}{\rho}_{i(1)}B_{i(1)}}{b_{i(1)} + p_{(1)}B_{i(1)}}\frac{\partial G_{(1)}}{\partial\overset{0}{\rho}_{i(1)}}; \quad Q_{2(1)} = \frac{\partial G_{(1)}}{\partial p_{(1)}} - \frac{p_{(1)}}{\rho_{(1)}}\sum_{i=1}^{m_1-1}\frac{\overset{0}{\rho}_{i(1)}}{b_{i(1)} + p_{(1)}B_{i(1)}}\frac{\partial G_{(1)}}{\partial\overset{0}{\rho}_{i(1)}};$$

$$G_{(1)}\left(p_{(1)}, \rho_{(1)}, \overset{0}{\rho}_{1(1)}, ..., \overset{0}{\rho}_{m_1-1}\right) = \frac{b_{m_1} + p_{(1)}\left[1 + B_{m_1} - \dfrac{\rho_{(1)}}{\rho_{0(1)}}\displaystyle\sum_{i=1}^{m_1-1}\frac{\alpha_{i0(1)}\overset{0}{\rho}_{i0(1)}\,(b_{im_1} + p_{(1)}B_{im_1})}{\overset{0}{\rho}_{i(1)}\,(b_{i(1)} + p_{(1)}B_{i(1)})}\right]}{\rho_{(1)}\left[B_{m_1} + \dfrac{\rho_{(1)}}{\rho_{0(1)}}\displaystyle\sum_{i=1}^{m_1-1}\frac{\alpha_{i0(1)}\overset{0}{\rho}_{i0(1)}\,(b_{m_1}B_{i(1)} - b_{i(1)}B_{m_1})}{\overset{0}{\rho}_{i(1)}\,(b_{i(1)} + p_{(1)}B_{i(1)})}\right]}.$$

Furthermore, the integrals

$$(u_{(1)} - \xi)^2 = G_{(1)}\left(p_{(1)}, \rho_{(1)}, \overset{0}{\rho}_{1(1)}, ..., \overset{0}{\rho}_{m_1-1}\right); \tag{21}$$

$$\alpha_{i(1)} = \alpha_{i0(1)}\frac{\rho_{(1)}\overset{0}{\rho}_{i0(1)}}{\rho_{0(1)}\overset{0}{\rho}_{i(1)}}, \quad i = 1, ..., m_1 - 1; \tag{22}$$

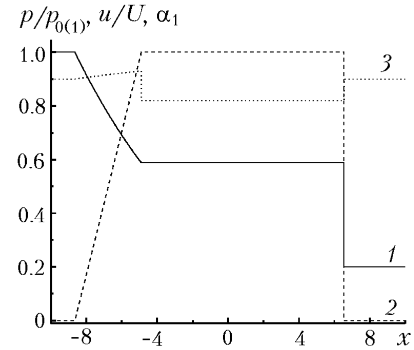

Fig. 2. Dependences of the parameters for the configuration of flow with a rarefaction wave and a shock: 1) $p(x)/p_{0(1)}$, 2) $u(x)/U$, and 3) $\alpha_{(1)}(x)$.

$$\alpha_{j(1)} = \alpha_{j0(1)} \frac{\rho_{(1)}}{\rho_{0(1)}} , \quad j = m_1 + 1, ..., n_1 . \tag{23}$$

are true for flow in the rarefaction wave. If some of the compressible fractions in the mixture are ideal gases for which $b_k = 0$, the heat flux equations for them yield (after their integration)

$$\overset{0}{\rho}_{k(1)} = \overset{0}{\rho}_{k0(1)} \frac{p_{(1)}}{p_{0(1)}} \left(\frac{\rho_{0(1)}}{\rho_{(1)}} \right)^{\frac{1}{B_{k(1)}}} , \tag{24}$$

which diminishes the number of differential equations in system (20).

System (20) is integrated from the initial values $p_{(1)} = p_{0(1)}$, $\rho_{(1)} = \rho_{0(1)}$, and $u_{(1)} = u_{0(1)}$ for $\xi = \xi_{0(1)}$ to a certain $\xi_{*(1)}$ value corresponding to such a pressure at which the velocities calculated from (20) and (3)–(6) are coincident. In so doing, we will ensure the observance of the condition of conjugation of the flow parameters on the contact discontinuity

$$p_{(1)} (\xi_{*(1)}) = p_{+(2)} = P , \quad u_{(1)} (\xi_{*(1)}) = u_{+(2)} = U . \tag{25}$$

The value of $\xi_{0(1)}$, in accordance with formula (21), is determined from the relation

$$\xi_{0(1)} = u_{0(1)} - c_{0(1)} ,$$

where $c_{0(1)}$ is the velocity of sound in the mixture, which is calculated from the formula

$$c_{0(1)} = \left[\frac{b_{m_1} + p_{0(1)} \left(1 + B_{m_1} - \sum_{i=1}^{m_1 - 1} \frac{\alpha_{i0(1)} (b_{im_1} + p_{0(1)} B_{im_1})}{b_{i(1)} + p_{0(1)} B_{i(1)}} \right)}{\rho_{0(1)} \left(B_{m_1} + \sum_{i=1}^{m_1 - 1} \frac{\alpha_{i0(1)} (b_{m_1} B_{i(1)} - b_{i(1)} B_{m_1})}{b_{i(1)} + p_{0(1)} B_{i(1)}} \right)} \right]^{1/2} . \tag{26}$$

As an example of flow including a shock wave and a rarefaction wave, we consider the problem on disintegration of an arbitrary discontinuity with the following initial data: $p_{0(1)} = 5.0 \cdot 10^5$ Pa, $p_{0(2)} = 10^5$ Pa, $u_{0(1)} = u_{0(2)} = 0$, $\alpha_{10(1)} = \alpha_{10(2)} = 0.9$, $\overset{0}{\rho}_{10(1)} = \overset{0}{\rho}_{10(2)} = 1.19$ kg/m^3, and $\overset{0}{\rho}_{20(1)} = \overset{0}{\rho}_{20(2)} = 1000$ kg/m^3. The constituent components of the mixture are the same as in the first example. In the calculations, we have obtained the following results: $P = 2.938 \cdot 10^5$ Pa, $U = 29.293$ m/sec, $D_{(2)} = 65.43$ m/sec, $\alpha_{1(2)} = 0.819$, $\xi_{0(1)} = -86.444$, and $\xi_{*(1)} = -48.94$. From these data, we easily reproduce the distribution of the mixture's parameters in the physical plane (x, t). In particular, Fig. 2 plots $p(x)/p_{0(1)}$, $u(x)/U$, and $\alpha_1(x)$ by the instant of time t = 0.1 sec after the discontinuity disintegration.

Two Rarefaction Waves. We consider the following possible configuration of discontinuity disintegration, namely, flow with two centered rarefaction waves. Relations (20)–(24) are true for the "left-hand" rarefaction wave. The solution for the "right-hand" wave can be obtained from the system

$$\frac{dp_{(2)}}{d\xi} = -\frac{2\rho_{(2)}\,(u_{(2)}-\xi)^3}{(u_{(2)}-\xi)^2\,(2+\rho_{(2)}Q_{1(2)})+\rho_{(2)}Q_{2(2)}}\,,\quad \frac{du_{(2)}}{d\xi} = \frac{2\,(u_{(2)}-\xi)^2}{(u_{(2)}-\xi)^2\,(2+\rho_{(2)}Q_{1(2)})+\rho_{(2)}Q_{2(2)}}\,,$$

$$\frac{d\rho_{(2)}}{d\xi} = -\frac{2\rho_{(2)}\,(u_{(2)}-\xi)}{(u_{(2)}-\xi)^2\,(2+\rho_{(2)}Q_{1(2)})+\rho_{(2)}Q_{2(2)}}\,, \tag{27}$$

$$\frac{d\overset{0}{\rho}_{i(2)}}{d\xi} = -\frac{2\overset{0}{\rho}_{i(2)}\,(u_{(2)}-\xi)\,[\rho_{(2)}B_{i(2)}\,(u_{(2)}-\xi)^2-p_{(2)}]}{(b_{i(2)}+p_{(2)}B_{i(2)})\,[(u_{(2)}-\xi)^2\,(2+\rho_{(2)}Q_{1(2)})+\rho_{(1)}Q_{2(2)}]}\,,\quad i=1,\,...,\,m_2-1\,.$$

Here we have

$$Q_{1(2)} = \frac{\partial G_{(2)}}{\partial p_{(2)}} + \sum_{i=1}^{m_2-1}\frac{\overset{0}{\rho}_{i(2)}B_{i(2)}}{b_{i(2)}+p_{(2)}B_{i(2)}}\,\frac{\partial G_{(2)}}{\partial\overset{0}{\rho}_{i(2)}}\,;\quad Q_{2(2)} = \frac{\partial G_{(2)}}{\partial p_{(2)}} - \frac{p_{(2)}}{\rho_{(2)}}\sum_{i=1}^{m_2-1}\frac{\overset{0}{\rho}_{i(2)}}{b_{i(2)}+p_{(2)}B_{i(2)}}\,\frac{\partial G_{(2)}}{\partial\overset{0}{\rho}_{i(2)}}\,;$$

$$G_{(2)}\left(p_{(2)},\,\rho_{(2)},\,\overset{0}{\rho}_{1(2)},\,...,\,\overset{0}{\rho}_{m_2-1}\right) = \frac{b_{m_2}+p_{(2)}\left[1+B_{m_2}-\dfrac{\rho_{(2)}}{\rho_{0(2)}}\displaystyle\sum_{i=1}^{m_2-1}\frac{\alpha_{i0(2)}\overset{0}{\rho}_{i0(2)}\,(b_{im_2}+p_{(2)}B_{im_2})}{\overset{0}{\rho}_{i(2)}\,(b_{i(2)}+p_{(2)}B_{i(2)})}\right]}{\rho_{(2)}\left[B_{m_2}+\dfrac{\rho_{(2)}}{\rho_{0(2)}}\displaystyle\sum_{i=1}^{m_2-1}\frac{\alpha_{i0(2)}\overset{0}{\rho}_{i0(2)}\,(b_{m_2}B_{i(2)}-b_{i(2)}B_{m_2})}{\overset{0}{\rho}_{i(2)}\,(b_{i(2)}+p_{(2)}B_{i(2)})}\right]}\,.$$

Systems (20) and (27) are integrated from the initial values $p_{(1)} = p_{0(1)}$, $\rho_{(1)} = \rho_{0(1)}$, and $u_{(1)} = u_{0(1)}$ and $p_{(2)} = p_{0(2)}$, $\rho_{(2)} = \rho_{0(2)}$, and $u_{(2)} = u_{0(2)}$ for $\xi = \xi_{0(1)}$ and $\xi = \xi_{0(2)}$ respectively to certain $\xi_{*(1)}$ and $\xi_{*(2)}$ values that are selected so as to ensure the observance of the conditions of conjugation of the flow parameters on the contact discontinuity:

$$p_{(1)}\,(\xi_{*(1)}) = p_{(2)}\,(\xi_{*(2)}) = P\,,\quad u_{(1)}\,(\xi_{*(1)}) = u_{(2)}\,(\xi_{*(2)}) = U\,. \tag{28}$$

The value of $\xi_{0(2)}$ in accordance with the integral of the system $(u_{(2)}-\xi)^2 = G_{(2)}(p_{(2)}, \rho_{(2)}, \overset{0}{\rho}_{1(2)}, ..., \overset{0}{\rho}_{m_2-1})$ is determined from the relation

$$\xi_{0(2)} = u_{0(2)} + c_{0(2)}\,,$$

where

$$c_{0(2)} = \left[\frac{b_{m_2}+p_{0(2)}\left(1+B_{m_2}-\displaystyle\sum_{i=1}^{m_2-1}\frac{\alpha_{i0(2)}\,(b_{im_2}+p_{0(2)}B_{im_2})}{b_{i(2)}+p_{0(2)}B_{i(2)}}\right)}{\rho_{0(2)}\left(B_{m_2}+\displaystyle\sum_{i=1}^{m_2-1}\frac{\alpha_{i0(2)}\,(b_{m_2}B_{i(2)}-b_{i(2)}B_{m_2})}{b_{i(2)}+p_{0(2)}B_{i(2)}}\right)}\right]^{1/2}\,.$$

There can be one more limiting configuration where the vacuum zone where the pressure is equal to zero is located between two rarefaction waves. The corresponding systems of differential equations (20) and (27) are integrated from $\xi_{0(i)}$ (i = 1 and 2) to such $\xi_{*(i)}$ values for which the pressures obtained from (20) and (27) vanish.

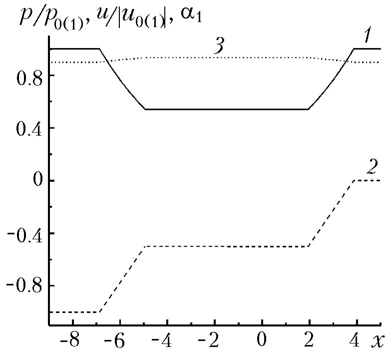

Fig. 3. Dependences of the parameters for the configuration of flow with two rarefaction waves: 1) $p(x)/p_{0(1)}$, 2) $u(x)/u_{0(1)}$, and 3) $\alpha_{(1)}(x)$.

Figure 3 plots $p(x)/p_{0(1)}$, $u(x)/|u_{0(1)}|$, and $\alpha_1(x)$ by the instant of time $t = 0.1$ sec for the initial data $p_{0(1)} = p_{0(2)} = 10^5$ Pa, $u_{0(1)} = -30$ m/sec, $u_{0(2)} = 0$, $\alpha_{10(1)} = \alpha_{10(2)} = 0.9$, $\rho^0_{10(1)} = \rho^0_{10(2)} = 1.19$ kg/m^3, and $p^0_{20(1)} = \rho^0_{20(2)} = 1000$ kg/m^3. In this case the regime of flow with two rarefaction waves is realized in discontinuity disintegration. The algorithm of calculation of this configuration involves minimization of the functional

$$F(\xi_{*(1)}, \xi_{*(2)}) = |p(\xi_{*(1)}) - p(\xi_{*(2)})| + |u(\xi_{*(1)}) - u(\xi_{*(2)})|$$

in the space of solutions of the system of equations (20) and (27). Computations are carried out with the standard procedures of the Mathcad package. In the calculations, we have obtained the following results: $\xi_{*(1)} = -49.518$, $\xi_{*(2)} = 19.488$, $P = 5.4 \cdot 10^4$ Pa, and $U = -15.01$ m/sec. From these data, we have reproduced the corresponding plots of Fig. 3.

NOTATION

c, velocity of sound in the mixture, m/sec; c_{*i}, constant of the equation of state, m/sec; D, velocity of the shock-wave front, m/sec; m, number of compressible fractions in the mixture; n, total number of fractions in the mixture; p, pressure, Pa; P, pressure at the contact boundary, Pa; t, time, sec; u, velocity, m/sec; U, velocity at the contact boundary, m/sec; x, spatial variable, m; α_i, volume fraction of the ith component of the mixture; A, volume fraction at the contact boundary; γ_i, constant of the equation of state; ε, specific internal energy, m^2/sec^2; ξ, self-similar variable; ρ, density of the mixture, kg/m^3; ρ^0_i, true density of the ith fraction, kg/m^3; ρ_i, reduced density of the ith component, kg/m^3; ρ_{*i}, constant of the equation of state, kg/m^3. Subscripts: 0, in an unperturbed medium; (1) and (2), for the parameters of the mixture to the "left" and to the "right" of the contact discontinuity; "−" and "+", for the parameters of the mixture in the region of contact discontinuity that refer to the values to its "left" and "right."

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